IDS 702: Module 3.5

Proportional odds model

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Ordinal responses

- Suppose the categories of our response variable has a natural ordering.
- Let's use data from Example 6.2.2 from Alan Agresti's An Introduction to Categorical Data Analysis, Second Edition to demonstrate this.
- This data is from a General Social Survey. Clearly, political ideology has a five-point ordinal scale, ranging from very liberal to very conservative.

		Political Ideology				
		Very Liberal	Slightly Liberal	Moderate	Slightly Conservative	Very Conservative
Female	Democratic	44	47	118	23	32
	Republican	18	28	86	39	48
Male	Democratic	36	34	53	18	23
	Republican	12	18	62	45	51

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Cumulative logits

- When we have ordinal response with categories $1, 2, \ldots, J$, we still want to estimate

$$\Pr[y_i = 1|\boldsymbol{x}_i] = \pi_{i1},\ \Pr[y_i = 2|\boldsymbol{x}_i] = \pi_{i2},\ \ldots,\ \Pr[y_i = J|\boldsymbol{x}_i] = \pi_{iJ}.$$

- However, we need to use models that can reflect the ordering

$$\Pr[y_i \leq 1|\boldsymbol{x}_i] \leq \Pr[y_i \leq 2|\boldsymbol{x}_i] \leq \ldots \leq \Pr[y_i \leq J|\boldsymbol{x}_i] = 1.$$

Notice that the ordering of probabilities is not for the actual marginal probabilities, but rather the cumulative probabilities.

- The multinomial logistic regression does not enforce this.
- Instead, we can focus on building models for the cumulative logits, that is, models for

$$\log\left(\frac{\Pr[y_i \leq j|\boldsymbol{x}_i]}{\Pr[y_i > j|\boldsymbol{x}_i]}\right) = \log\left(\frac{\pi_{i1} + \pi_{i2} + \ldots + \pi_{ij}}{\pi_{i(j+1)} + \pi_{i(j+2)} + \ldots + \pi_{iJ}}\right), \quad j = 1, \ldots, J-1.$$

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- This leads us to the proportional odds model, written as:

$$\log\left(\frac{\Pr[y_i \leq j|\boldsymbol{x}_i]}{\Pr[y_i > j|\boldsymbol{x}_i]}\right) = \beta_{0j} + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip}, \quad j = 1, \ldots, J-1.$$

  There is no need for a model for $\Pr[y_i \leq J]$ *since it is necessarily equal to 1.*

- Notice that this model looks like a binary logistic regression in which we combine the first j categories to form a single category (say 1) and the remaining categories to form a second category (say 0).
- Since β_0 is the only parameter indexed by j, the $J-1$ logistic regression curves essentially have the same shapes but different "intercepts".
- That is, the effect of the predictors is identical for all $J-1$ cumulative log odds. This is therefore, a more parsimonious model (both in terms of estimation and interpretation) than the multinomial logistic regression, when it fits the data well.

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- The probabilities we care about are quite easy to extract, since each

$$\Pr[y_i = j|\boldsymbol{x}_i] = \Pr[y_i \leq j|\boldsymbol{x}_i] - \Pr[y_i \leq j-1|\boldsymbol{x}_i], \quad j = 2, \ldots, J,$$

  with $\Pr[y_i \leq 1|\boldsymbol{x}_i] = \Pr[y_i = 1|\boldsymbol{x}_i]$.

- Let's focus first on a single continuous predictor, that is,

$$\log\left(\frac{\Pr[y_i \leq j|x_i]}{\Pr[y_i > j|x_i]}\right) = \beta_{0j} + \beta_1 x_{i1}, \quad j = 1, \ldots, J-1.$$

  Here, $\beta_1 > 0$, actually means that a 1 unit increase in x makes the larger values of Y less likely.

- This can seem counter-intuitive, thus, many books and software packages (including the `polr` function in R) often write

$$\log\left(\frac{\Pr[y_i \leq j|x_i]}{\Pr[y_i > j|x_i]}\right) = \beta_{0j} - \beta_1 x_{i1}, \quad j = 1, \ldots, J-1$$

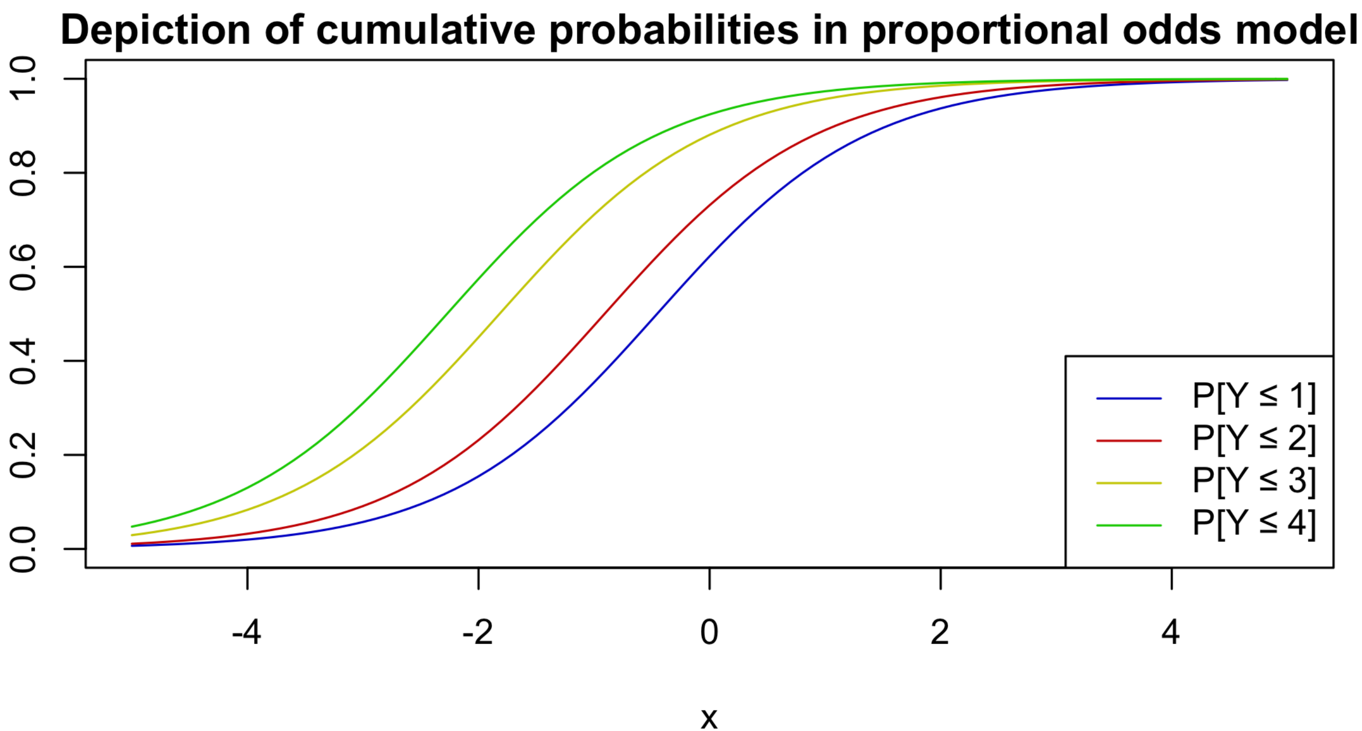
  instead. We will stick with this representation.

Proportional odds model

- Suppose we have $J = 5$, $\beta_1 = 1.1$, and $(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{04}) = (0.5, 1, 2, 2.5)$ in the first representation

$$\log\left(\frac{\Pr[y_i \leq j|x_i]}{\Pr[y_i > j|x_i]}\right) = \beta_{0j} + \beta_1 x_{i1}, \quad j = 1, \ldots, 4,$$

the cumulative probabilities would look like:



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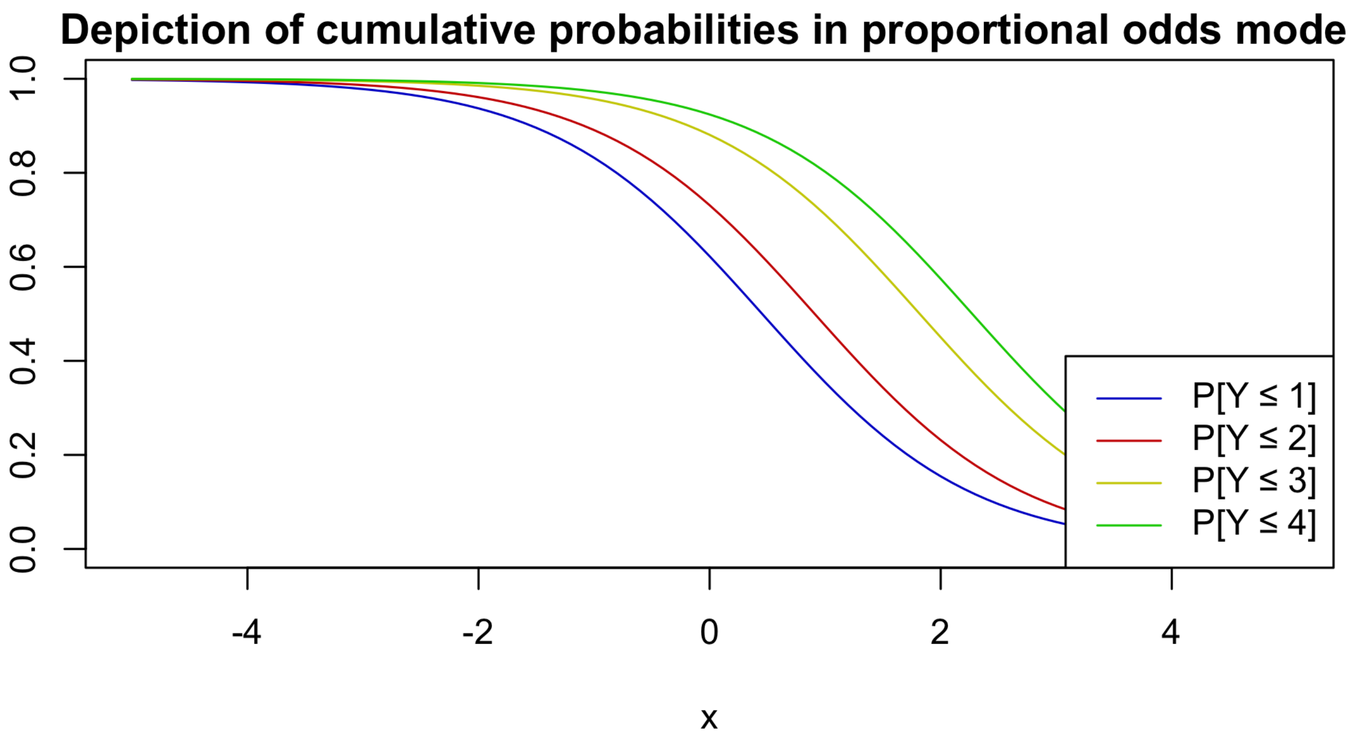
Proportional odds model

- But with $J = 5$, and the same values $\beta_1 = 1.1$, and $(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{04}) = (0.5, 1, 2, 2.5)$ in the second representation

$$\log\left(\frac{\Pr[y_i \leq j|x_i]}{\Pr[y_i > j|x_i]}\right) = \beta_{0j} - \beta_1 x_{i1}, \quad j = 1, \ldots, 4,$$

the cumulative probabilities would look like:



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- Take our example on political ideology for instance. Suppose we fit the model

$$\log\left(\frac{\Pr[\text{ideology}_i \le j|x_i]}{\Pr[\text{ideology}_i > j|x_i]}\right) = \beta_{0j} - \beta_1 x_{i1}, \quad j = 1, \ldots, 4,$$

  where x is an indicator variable for political party, with $x = 1$ for Democrats and $x = 0$ for Republicans.

- Then,

  - For any j, β_1 is the log-odds of a Democrat, when compared to a Republican, of being more conservative than j compared to being more liberal than j.

  - For any j, e^{β_1} is the odds of a Democrat, when compared to a Republican, of being more conservative than j compared to being more liberal than j.

- If $\beta_1 > 0$, a Democrat's response .is more likely than a Republican's response to be in the conservative direction than in the liberal direction.

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What's next?

Move on to the readings for the next module!